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Analytic study of temperature solutions due to gamma-type moving point-heat sources

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Abstract—A closed-form model for the computation of temperature distribution in an infinitely extended isotropic body with Gamma-type moving point-heat sources is discussed. The temperature solutions for these moving sources are discussed for time-dependent sources of the forms: (i) $\dot{Q}_1(t) = \dot{Q}_0 \exp(-\lambda t)$, and (ii) $\dot{Q}_2(t) = \dot{Q}_0(t/t^*) \exp(-\lambda t)$, where λ is a real number and t^* characterizes the limiting time. The reduced (or dimensionless) temperature solutions are presented in terms of the generalized representation of an incomplete Gamma function $I_\alpha(b, x)$, which is also expressed by complementary error functions. It is also demonstrated that the present analysis covers the classical temperature solution of a constant strength source under quasi-steady-state situations.

1. INTRODUCTION

THE HEAT released by a moving source whose extent can be neglected as compared with the dimensions of the surroundings of interest, is important for analyzing several technological processes. These transient heat conduction problems originating from a moving source of heat may be due to (i) sliding friction, (ii) machining, and (iii) numerous metal-treating operations such as welding, casting, quenching, and flame-hardening. It should be noted that these problems are typically grouped in the following three categories: (a) the moving point-heat source; (b) the moving line-heat source; and (c) the moving plane-heat source.

Spraragen and Claussen [1] have reviewed the early approximate theory of moving heat sources, while somewhat exact formulation is due to Rosenthal [2] which is also discussed in Handbook of Heat Transfer Fundamentals [3]. According to Rosenthal's theory, an excitation heat source moves through a fixed region (x, y, z) with constant unidirectional velocity u (parallel to x), and the temperature-response function $T(x, y, z, t)$ must satisfy the transient heat conduction equation. He has demonstrated that a solution for 'T' is simplified by transforming to a moving coordinate ξ whose origin coincides with the heat source and moves at the source velocity u. This scheme has also been exploited by Carslaw and Jaeger [4] and Grigull and Sandner [5] to discuss quasi-steady-state solution of moving point-heat sources of constant strength. The quasi-steady-state solution of a moving Gaussian-type source has been performed by Cline and Anthony [6] and Mazumder and Steen [7].

It should be noted that the closed-form solutions for the constant as well as the general, time-dependent, moving heat sources in an infinite solid are not available in the literature [1–7]. The objective of this paper is to present closed-form solutions for the time-dependent moving Gamma-type point-heat sources.

2. MATHEMATICAL FORMULATION

We consider a time-dependent point heat source in a homogeneous and isotropic body of an infinite extension, moving with a velocity u along the x direction. The solution of transient, heat conduction equation in the infinite solid due to the point-heat source of strength $\dot{Q}(t)$ may be obtained by adding the contributions of an infinite number of small instantaneous sources placed one behind the other at infinitely small intervals of time, along the x direction. If the quantity of heat released from time $t = \tau$ to $t = \tau + \mathrm{d}\tau$ is $\dot{Q}(\tau)\,\mathrm{d}\tau$, then the temperature field formed about a moving point heat source to which we have attached the origin of the rectangular coordinate system is given by [4, 5]

$$T(x, y, z, t) = \frac{1}{(4\pi\alpha)^{3/2}\rho C_p} \int_0^t \dot{Q}(\tau) \times \exp\left[-\frac{[x-u(t-\tau)]^2+y^2+z^2}{4\alpha(t-\tau)}\right] \frac{\mathrm{d}\tau}{(t-\tau)^{3/2}}. \quad (1)$$

The abbreviation $r^2 = x^2+y^2+z^2$ and the substitution

$$\phi = \frac{r^2}{4\alpha(t-\tau)}, \quad \text{and hence} \quad \sqrt{\frac{4\alpha}{r^2}}\frac{\mathrm{d}\phi}{\phi^{1/2}} = \frac{\mathrm{d}\tau}{(t-\tau)^{3/2}}$$

allows us to reduce equation (1) to the form

$$T(x, r, t) = \frac{e^{(ux/2\alpha)}}{4\pi^{3/2}kr} \int_{(r^2/4\alpha t)}^{\infty} \dot{Q}(t - r^2/4\alpha\phi) \times \exp\left[-\phi - \left(\frac{ut}{r}\right)^2 \left(\frac{r^2}{4\alpha t}\right)^2 \phi^{-1}\right] \frac{\mathrm{d}\phi}{\phi^{1/2}}. \quad (2)$$

3. SOME CLOSED-FORM SOLUTIONS

In this section, we use equation (2) to discuss the temperature field formed about moving, Gamma-type point-heat sources of the following forms:

NOMENCLATURE

- C_p specific heat at constant pressure [KJ kg^{-1} K^{-1}]
- Fo Fourier number, $\alpha t/x^2$
- I generalized Gamma function
- k thermal conductivity [W m^{-1} K^{-1}]
- $\dot{Q}$ heat rate [W]
- r distance from the heat source [m]
- t time [s]
- T temperature [K]
- u source velocity [m sec^{-1}]
- V reduced velocity, ut/r.

Greek symbols
- α thermal diffusivity, $k/\rho C_p$ [m^2 s^{-1}]
- β dimensionless parameter, $V^2/4Fo-\tau$
- Γ Gamma function
- θ reduced (or dimensionless) temperature
- ρ density [kg m^{-3}]
- τ reduced time constant, λt.

Subscripts
- 1 exponential-type, moving-point source
- 11 quasi-steady-state constant source
- 2 pulse-type, moving-point source
- 21 linearly increasing point source
- 22 quasi-steady-state linear source.

Superscript
- * limiting value.

$$\dot{Q}_1 = \dot{Q}_0 \exp(-\lambda t), \quad \text{and} \tag{3}$$

$$\dot{Q}_2 = \dot{Q}_0(t/t^*) \exp(-\lambda t), \tag{4}$$

where t^* denotes a time-interval that characterizes the moving source independently from the thermo-physical properties of the medium, and λ is any real number.

3.1. *The exponential-type point-heat source*

We note that an exponential-type heat source given by equation (3), when substituted in equation (2), results in

$$T_1(x,r,t) = \frac{\dot{Q}_1(t)\,e^{(ux/2\alpha)}}{4\pi^{3/2}kr}\int_{r^2/4\alpha t}^{\infty}\phi^{1/2-1} \times \exp\left\{-\phi-\left[\left(\frac{ut}{r}\right)^2\left(\frac{r^2}{4\alpha t}\right)-\lambda t\right] \times \left(\frac{r^2}{4\alpha t}\right)\phi^{-1}\right\}\mathrm{d}\phi, \tag{5}$$

which can easily be simplified by using equation (A.1) as

$$T_1(x,r,t) = \frac{e^{(ux/2\alpha)}\dot{Q}_1(t)}{4\pi^{3/2}kr} \times I_{1/2}\left[\left(\frac{ut}{r}\right)^2\left(\frac{r^2}{4\alpha t}\right)-\lambda t, \left(\frac{r^2}{4\alpha t}\right)\right]. \tag{6}$$

It is convenient to introduce the above solution in the dimensionless form as

$$\theta_1 = I_{1/2}(\beta, 1/4Fo), \tag{7}$$

where

$$\theta_1 = \frac{4\pi^{3/2}krT_1(x,r,t)}{e^{(ux/2\alpha)}\dot{Q}_1(t)}, \tag{8}$$

$$\beta = V^2/4Fo-\tau, \tag{9}$$

$$V = ut/r, \tag{10}$$

$$Fo = \alpha t/r^2, \tag{11}$$

$$\tau = \lambda t. \tag{12}$$

We note that equation (7) can be expressed in terms of complementary error function by using equation (A.3) as

$$\theta_1 = \frac{\sqrt{\pi}}{2}[\exp(-\sqrt{\beta/Fo})\,\mathrm{Erfc}(1/2\sqrt{Fo}-\beta) + \exp(\sqrt{\beta/Fo})\,\mathrm{Erfc}(1/2\sqrt{Fo}+\beta)]. \tag{13}$$

For $\lambda = 0$, we note that equation (3) represents a constant strength heat source. The substitution of this value of λ in equations (9), (12) yields

$$\beta = \beta_0 = V^2/4Fo. \tag{14}$$

It is of interest to recover the temperature solution of a constant strength source under quasi-steady-state situation, i.e. $t \to \infty$. This can be obtained from equation (13) when $\beta = \beta_0 \to \infty$ and $1/4Fo \to 0$. Using equations (A.5)–(A.7) we get

$$\theta_{11} = \pi^{1/2}\exp(-\sqrt{\beta_0/Fo}). \tag{15}$$

On using equations (8), (10), (11) and (14), we can simplify equation (15) to give

$$T(x,y,z) = \frac{\dot{Q}_1}{4\pi kr}\exp\left[\frac{-u(r-x)}{2\alpha}\right], \tag{16}$$

which is the same steady-state solution as that reported in Carslaw and Jaeger [4] and Grigull and Sandner [5].

3.2. *The pulse-type point-heat source*

We note that a pulse-type heat source given by equation (4), when substituted in equation (2), results in

$$T_2(x,r,t) = \frac{\dot{Q}_2(t)\,e^{(ux/2\alpha)}}{4\pi^{3/2}kr}$$

$$\times \int_{1/4Fo}^{\infty} [\exp(-\phi-\beta/4Fo\,\phi)\phi^{1/2-1}$$

$$-(1/4Fo)\exp(-\phi-\beta/4Fo\,\phi)\phi^{-1/2-1}] \quad (17)$$

which can easily be integrated by using equation (A.1) as

$$\theta_2 = [I_{1/2}(\beta, 1/4Fo) - (1/4Fo)I_{-1/2}(\beta, 1/4Fo)], \quad (18)$$

where

$$\theta_2 = \frac{4\pi^{3/2}krT_2(x,r,t)}{e^{(ux/2\alpha)}\dot{Q}_2(t)}$$

and β, Fo are the same as that described earlier in equations (9) and (11).

It should be noted that the dimensionless (or reduced) temperature expression given by equation (18) can easily be expressed in terms of the (well-known) complementary error function by using equations (A.3)–(A.4) as

$$\theta_2 = \frac{\sqrt{\pi}}{2}[\exp(-\sqrt{\beta/Fo})\,\mathrm{Erfc}\,(1/2\sqrt{Fo}-\beta)$$

$$+\exp(\sqrt{\beta/Fo})\,\mathrm{Erfc}\,(1/2\sqrt{Fo}+\beta)]$$

$$-\frac{2\sqrt{Fo}}{\sqrt{\beta}}[\exp(-\sqrt{\beta/Fo})\,\mathrm{Erfc}\,(1/2\sqrt{Fo}-\beta)$$

$$-\exp(\sqrt{\beta/Fo})\,\mathrm{Erfc}\,(1/2\sqrt{Fo}+\beta)]. \quad (19)$$

We note that for $\lambda = 0$, equation (4) represents a point-source of linearly increasing strength. The substitution of this value of λ in equation (18) yields

$$\theta_{21} = [I_{1/2}(\beta_0, 1/4Fo) - (1/4Fo)I_{-1/2}(\beta_0, 1/4Fo)], \quad (20)$$

which is the same as that given by equation (19), with $\beta = \beta_0$.

The quasi-steady-state representation of equation (20) by using equations (A.5)–(A.6) can be expressed as

$$\theta_{22} = \pi^{1/2}(1 - 1/2\sqrt{\beta_0 Fo})\exp[-\sqrt{\beta_0/Fo}]. \quad (21)$$

Notice that as $t \to \infty$, $\sqrt{\beta_0 Fo} \to \infty$, thus

$$\theta_{22} \approx \pi^{1/2}\exp(-\sqrt{\beta_0/Fo}). \quad (22)$$

The graphical representation of equations (13) and (19) is shown in Figs. 1–3, respectively. In Figs. 1 and 2, the reduced temperature θ is plotted vs the dimensionless parameter β, for various values of the reduced time parameter Fo. We note that the reduced temperature plots are represented by characteristic Gaussian-type curves. It can be seen from these figures that for large values of β, the reduced temperature values approach the zero value. It should, however, be noted that the θ values for an exponential-type heat source (refer to Fig. 1) are relatively larger compared to the pulse-type (refer to Fig. 2), particularly at low values of the dimensionless parameter β. For example, at $\beta = 1.00$ and $Fo = 5.00$, the reduced temperatures θ_1, θ_2 are 1.1 and 0.90, respectively.

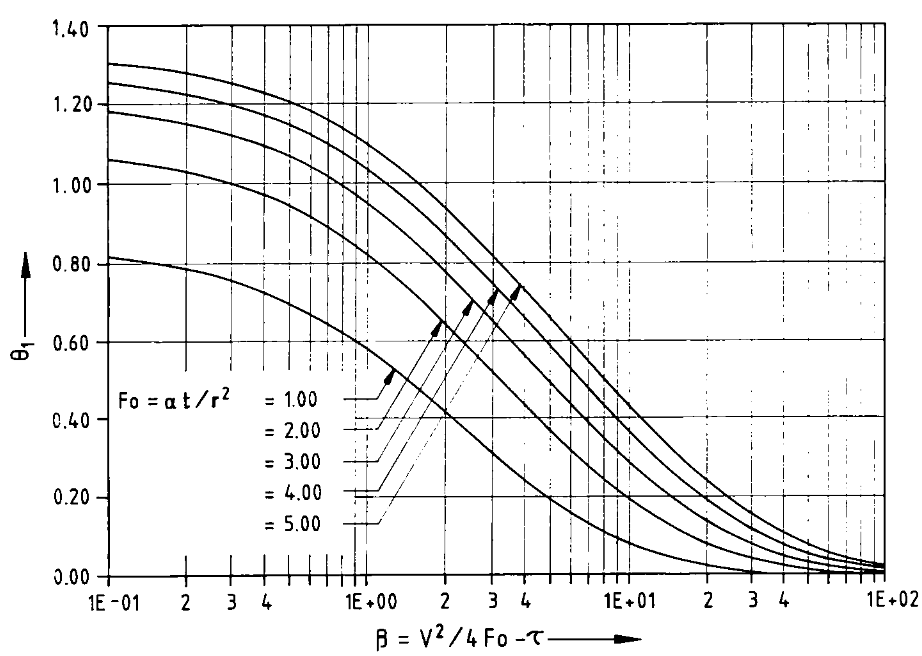

FIG. 1. Reduced temperature vs the dimensionless parameter from a moving point-heat source of an exponential-type strength of the form $\dot{Q}_0 \exp(-\lambda t)$.

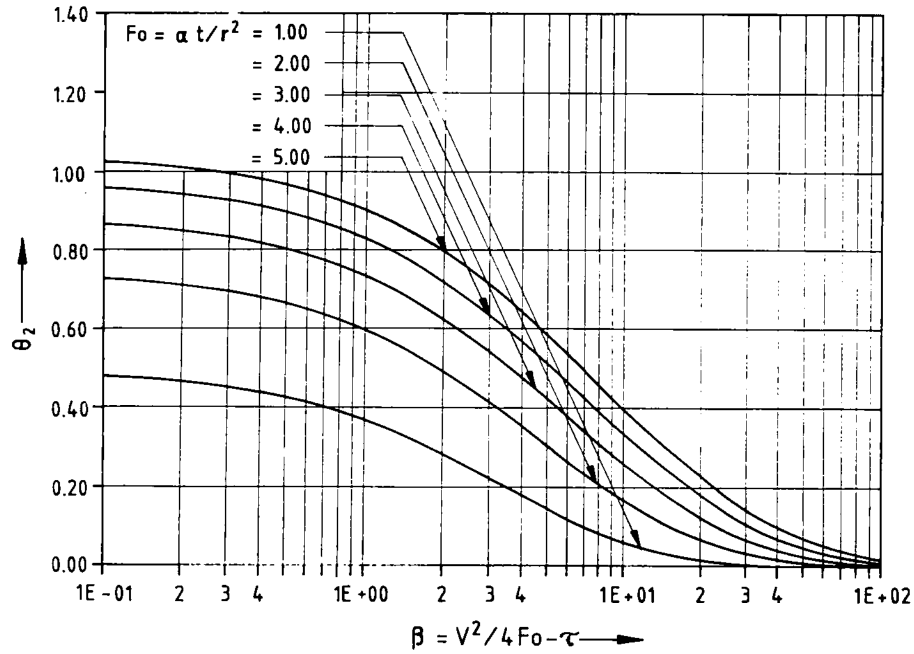

FIG. 2. Reduced temperature vs the dimensionless parameter from a moving point-heat source of a pulse-type strength of the form $\dot{Q}_0(t/t^*)\exp(-\lambda t)$.

The dimensionless parameter β, which is defined by equation (9) is presented in Fig. 3. In this figure, $(\beta+\tau)$ is plotted vs the reduced velocity V, for various values of the reduced time parameter. The $(\beta+\tau)$ plots shown in this figure are represented by parabolic-type curves. It should be noted that the τ value given by equation (12) may be defined as the reduced time parameter.

4. CONCLUDING REMARKS

The analytical solutions of temperature distributions due to time-dependent moving point-heat

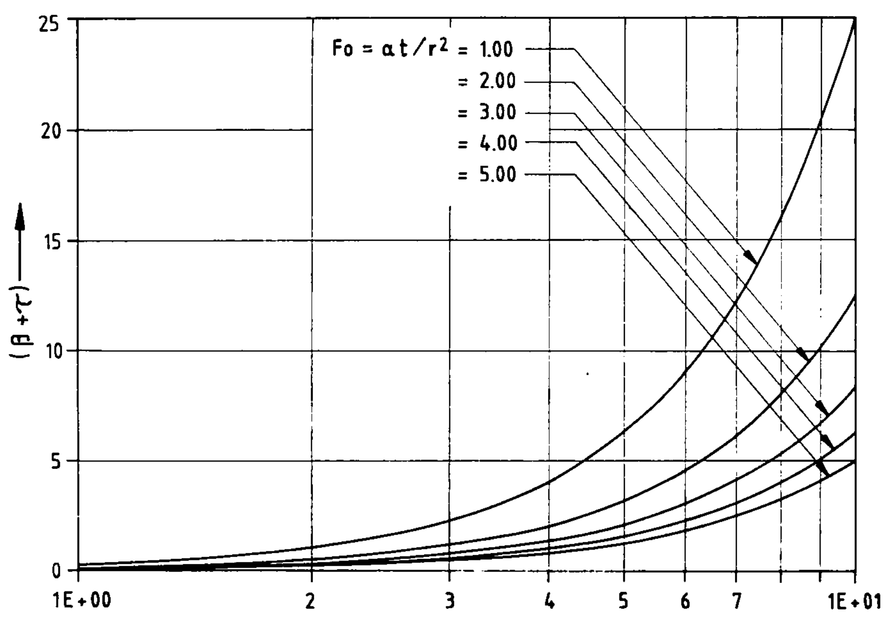

FIG. 3. The dimensionless parameter $(\beta+\tau)$ vs the reduced velocity.

sources are discussed for an extended, homogeneous, and the isotropic medium. The strength of moving heat sources considered in this paper is exponential- and pulse-type. All the closed-form temperature solutions are presented in terms of the reduced temperature (θ) as a function of the dimensionless parameter (β) and the reduced time parameter (Fo). Although, the quasi-steady-state solutions for the constant strength source are discussed in Carslaw and Jaeger [4] and Grigull and Sandner [5], the closed-form transient solutions for the moving point-heat source of constant strength are not available in the literature. The present analysis provides the closed-form solutions for the constant as well as the Gamma-type moving point-heat sources.

It should be noted that the time-dependent, continuously operating moving heat source of the form $\dot{Q}_0[a+b(t/t^*)]\exp(-\lambda t)$ may be considered as the most general Gamma-type heat source, in which equations (3) and (4) may be recovered by specializing the parameters a and b. The reduced temperature solutions due to such type of a heat source can now be expressed as

$$\theta_3 = a\theta_1 + b\theta_2. \tag{23}$$

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REFERENCES

1. W. Spraragen and G. E. Claussen, Temperature distribution during welding—A review of the literature to January 1, 1937, *Welding J. Res. Sup.* **16,** 4–10 (September 1937).
2. D. Rosenthal, The theory of moving sources of heat and its application to metal treatments, *Trans. Am. Soc. Mech. Engrs.* **68,** 849–866 (1946).
3. W. M. Rohsenow, J. P. Hartnett and E. N. Ganic, *Handbook of Heat Transfer Fundamentals* (2nd Edn). McGraw-Hill, New York (1985).
4. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*. Oxford University Press, London (1959).
5. U. Grigull and H. Sandner, *Heat Conduction* (English Translation Ed. by J. Ketsin). Springer, Berlin (1984).
6. H. E. Cline and T. R. Anthony, Heat treating and melting material with a scanning laser or electron beam, *J. appl. Phys.* **48,** 3895–3900 (1977).
7. J. Mazumder and W. M. Steen, Heat transfer model for cw laser material processing, *J. appl. Phys.* **51,** 941–947 (1980).
8. G. Gasper, Personal correspondence with M. A. Chaudhry (October 1991).
9. M. A. Chaudhry and S. M. Zubair, Generalized incomplete gamma function with applications, paper submitted to *J. Comput. Appl. Maths.* (1992).

APPENDIX

It should be noted that the integral occurring in equations (5) and (17) can be represented as [8]

$$I_\alpha(b,x) = \int_x^\infty t^{\alpha-1}\,e^{-t-xb/t}\,dt, \tag{A.1}$$

which is considered as a generalized representation of an incomplete Gamma function [9], e.g. the above integral at $b = 0$ reduces to

$$I_\alpha(0,x) = \Gamma(\alpha,x) = \int_x^\infty t^{\alpha-1}\,e^{-t}\,dt. \tag{A.2}$$

In general, equation (A.1) belongs to the family of Weyl (fractional) integrals, which can be integrated for certain values of α to give the following results [9]

$$I_{1/2}(b,x) = \frac{\sqrt{\pi}}{2}[\exp(-2\sqrt{bx})\,\mathrm{Erfc}(\sqrt{x}-\sqrt{b}) + \exp(2\sqrt{bx})\,\mathrm{Erfc}(\sqrt{x}+\sqrt{b})], \tag{A.3}$$

$$I_{-1/2}(b,x) = \frac{\sqrt{\pi}}{2\sqrt{bx}}[\exp(-2\sqrt{bx})\,\mathrm{Erfc}(\sqrt{x}-\sqrt{b}) - \exp(2\sqrt{bx})\,\mathrm{Erfc}(\sqrt{x}+\sqrt{b})]. \tag{A.4}$$

The asymptotic value of $I_\alpha(b,x)$ when $x \to 0$ can be written as

$$I_\alpha(b,x) = 2(bx)^{\alpha/2}K_\alpha(2\sqrt{bx}) - x^\alpha e^{-x-b}, \quad x \to 0, \tag{A.5}$$

where K_α is the modified Bessel function of the second kind.

Letting $x \to 0$ and $b \to \infty$ in equation (A.5), we get

$$I_\alpha(b,x) \approx 2(bx)^{\alpha/2}K_\alpha(2\sqrt{bx}); \quad x \to 0 \quad \text{and } b \to \infty \tag{A.6}$$

where

$$K_{\pm 1/2}(x) = \frac{\pi^{1/2}}{\sqrt{2x}}\,e^{-x}. \tag{A.7}$$